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Absolute Continuity of Vitali–Hahn–Saks Measure Convergence Theorems

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In this paper, we prove the following improved Vitali–Hahn–Saks measure convergence theorem: Let $(L, 0, 1)$ be a Boolean algebra with the sequential completeness property, (G, τ) be an Abelian topological group, ν be a nonnegative finitely additive measure defined on L, $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of finitely additive s-bounded G-valued measures defined on L, too. If for each $a \in L$, $\{\mu_n(a)\}_{n\in\mathbf{N}}$ is a τ-convergent sequence, for each $n \in \mathbf{N}$, when $\{\nu(a_\alpha)\}_{\alpha\in\Lambda}$ convergent to 0, $\{\mu_n(a_\alpha)\}_{\alpha\in\Lambda}$ is τ-convergent, then when $\{\nu(a_\alpha)\}_{\alpha\in\Lambda}$ convergent to 0, $\{\mu_n(a_\alpha)\}_{\alpha\in\Lambda}$ are τ-convergent uniformly with respect to $n \in \mathbf{N}$.

KEY WORDS: measures; Boolean algebras; Vitali–Hahn–Saks theorem.

Let $(L, 0, 1)$ be a Boolean algebra, (G, τ) be an Abelian topological group, a mapping $\mu : L \to G$ is said to be a finitely additive measure if $a, b \in L$ with $a \wedge b = 0$, then $\mu(a \vee b) = \mu(a) + \mu(b)$. The measure μ is said to be *s-bounded* if for each disjoint sequence $\{a_n\}$ of $(L, 0, 1)$, $\{\mu(a_n)\}$ is τ-convergent to 0. Let $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of finitely additive s-bounded measures, if for each disjoint sequence $\{a_k\}$ of $(L, 0, 1)$, $\{\mu_n(a_k)\}$ are τ-convergent to 0 uniformly with respect to $n \in \mathbf{N}$, then $\{\mu_n : n \in \mathbf{N}\}$ is said to be *uniformly s-bounded*.

Brooks and Jewett (1970) proved the following famous Vitali–Hahn–Saks measure convergence theorem:

Theorem 1′. *Let $\mathcal{A}$ be a σ-algebra, $(X, ||.||)$ be a Banach space, ν be a nonnegative finitely additive measure defined on $\mathcal{A}$, $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of finitely additive s-bounded X-valued measures defined on $\mathcal{A}$, too. If for each $A \in \mathcal{A}$, $\{\mu_n(A)\}_{n\in\mathbf{N}}$ is a $||.||$-convergent sequence, for each $n \in \mathbf{N}$, $\lim_{\nu(A)\to 0} \mu_n(A) = 0$, then $\lim_{\nu(A)\to 0} \mu_n(A) = 0$ uniformly with respect to $n \in \mathbf{N}$.*

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That is, if for each $n \in \mathbf{N}$, μ_n is absolutely continuous with respect to ν, then $\{\mu_n\}_{n\in\mathbf{N}}$ are absolutely continuous with respect to ν uniformly for $n \in \mathbf{N}$.

Vitali–Hahn–Saks theorem has a series of important applications in measure theory and quantum logics (De Simone, 2000).

Now, we are interested in the following problem: If for each $n \in \mathbf{N}$, when $\{\nu(A_\alpha)\}_{\alpha\in\Lambda}$ convergent to 0, $\{\mu_n(a_\alpha)\}_{\alpha\in\Lambda}$ is $||.||$-convergent to e_n, then when $\{\nu(a_\alpha)\}_{\alpha\in\Lambda}$ convergent to 0, $\{\mu_n(a_\alpha)\}_{\alpha\in\Lambda}$ whether are τ-convergent to e_n uniformly with respect to $n \in \mathbf{N}$? That is, whether we can improve the absolute continuity of Vitali–Hahn–Saks theorem?

In this paper, by considering $L \times L$ and using the proof methods of Brooks–Jewett (Brooks and Jewett, 1970), we show that the answer is true.

Our main result is

Theorem 1. *Let $(L, 0, 1)$ be a Boolean algebra with the sequential completeness property, (G, τ) be an Abelian topological group, ν be a nonnegative finitely additive measure defined on L, $\{\mu_n : n \in \mathbf{N}\}$ be a sequence of finitely additive s-bounded G-valued measures defined on L, too. If for each $a \in L$, $\{\mu_n(a)\}_{n\in\mathbf{N}}$ is a τ-convergent sequence, for each $n \in \mathbf{N}$, when $\{\nu(a_\alpha)\}_{\alpha\in\Lambda}$ convergent to 0, $\{\mu_n(a_\alpha)\}_{\alpha\in\Lambda}$ is τ-convergent to e_n, then when $\{\nu(a_\alpha)\}_{\alpha\in\Lambda}$ convergent to 0, $\{\mu_n(a_\alpha)\}_{\alpha\in\Lambda}$ are τ-convergent to e_n uniformly with respect to $n \in \mathbf{N}$.*

Proof: If the conclusion is not true, there exists $\varepsilon > 0$ and sequences $\{n_k\}$, $\{\delta_k\}$, $\{a_k\}$ and $\{b_k\}$, and a τ-continuous group quasi-norm P such that $P(\mu_{n_{k+1}}(a_{k+1}) - \mu_{n_{k+1}}(b_{k+1})) > \varepsilon$, $\nu(a_{k+1}) < \delta_{k+1}$, $\nu(b_{k+1}) < \delta_{k+1}$, and $\nu(a) < \delta_{k+1}$, $\nu(b) < \delta_{k+1}$ implies that $P(\mu_{n_i}(a) - \mu_{n_i}(b)) < \frac{\varepsilon}{2^{k+3}}$ for $i \le k$. Without loss of generality, we may assume that $n_i = i$. So

$$P(\mu_{k+1}(a_{k+1}) - \mu_{k+1}(b_{k+1})) > \varepsilon, \tag{1}$$

$$P(\mu_j(a) - \mu_j(b)) < \frac{\varepsilon}{2^{k+3}}, \ j \le k, \ a \le a_{k+1}, \ b \le b_{k+1}. \tag{2}$$

Consider $L \times L = \{(c, d) : c \in L, d \in L\}$. Let $c_1 = a_2, d_1 = b_2$ and $i_1 = 2$. If there exists an $i_2 > 2$ such that $P(\mu_{i_2}(c_1 \wedge a_{i_2}) - \mu_{i_2}(d_1 \wedge b_{i_2})) > \frac{\varepsilon}{4}$, then let $(c_2, d_2) = (c_1 \wedge a'_{i_2}, d_1 \wedge b'_{i_2})$. If $(c_1, d_1), \ldots, (c_k, d_k)$ and $i_1, \ldots, i_k$ have been chosen and that there exists an $i_{k+1} > i_k$ such that $P(\mu_{i_{k+1}}(c_k \wedge a_{i_{k+1}}) - \mu_{i_{k+1}}(d_k \wedge b_{i_{k+1}})) > \frac{\varepsilon}{4}$, then let $(c_{k+1}, d_{k+1}) = (c_k \wedge a'_{i_{k+1}}, d_k \wedge b'_{i_{k+1}})$. Thus, we have

$$c_{k+1} \le c_k, \ d_{k+1} \le d_k, \ c_k \wedge c'_{k+1} = c_k \wedge a_{i_{k+1}}, \ d_k \wedge d'_{k+1} = d_k \wedge b_{i_{k+1}}. \tag{3}$$

It follows from (2) and (3) that

$$P(\mu_{i_{k+1}}(c_k \wedge c'_{k+1}) - \mu_{i_{k+1}}(d_k \wedge d'_{k+1})) > \frac{\varepsilon}{4}. \tag{4}$$

$$P(\mu_{i_k}(c_k \wedge c'_{k+1}) - \mu_{i_k}(d_k \wedge d'_{k+1})) < \frac{\varepsilon}{2^{k+3}}. \tag{5}$$

Now, we show that there exists a $(c_{k_0}, d_{k_0}) \in L \times L$ and an i_{k_0} such that for all $j > i_{k_0}$, $P(\mu_j(c_{k_0} \wedge a_j) - \mu_j(d_{k_0} \wedge b_j)) < \frac{\varepsilon}{4}$.

In fact, if not, we can obtain disjoint sequence $\{c_k \wedge c'_{k+1}\}$ and disjoint sequence $\{d_k \wedge d'_{k+1}\}$ in L which satisfy (4) and (5) for all $k \in \mathbf{N}$. Thus, we have

$$P((\mu_{i_{k+1}} - \mu_{i_k})(c_k \wedge c'_{k+1}) - (\mu_{i_{k+1}} - \mu_{i_k})(d_k \wedge d'_{k+1})) > \frac{\varepsilon}{8}, k = 1, 2, \ldots.$$

This contradicts the Theorem 1 of Junde and Zhihao (2003). Hence, there exists a $(c_{k_0}, d_{k_0}) \in L \times L$ and an i_{k_0} such that for all $j > i_{k_0}$, $P(\mu_j(c_{k_0} \wedge a_j) - \mu_j(d_{k_0} \wedge b_j)) < \frac{\varepsilon}{4}$.

Let $p_1 = i_{k_0}$, $(h_1, g_1) = (c_{k_0}, d_{k_0})$, $\mu_i^{(1)} = \mu_{p_1+i}$, $(a_i^{(1)}, b_i^{(1)}) = (a_{p_1+i} \wedge h'_1, b_{p_1+i} \wedge g'_1)$. It follows from (1), (2), and (5) easily that

$$P(\mu_1(h_1) - \mu_1(g_1)) < \frac{\varepsilon}{2^{1+3}} = \frac{\varepsilon}{16},$$

$$P(\mu_2(h_1) - \mu_2(g_1)) > \varepsilon - \frac{\varepsilon}{4}.$$

So

$$P((\mu_2 - \mu_1)(h_1) - (\mu_2 - \mu_1)(g_1)) > \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{16},$$

$$P\big(\mu_i^{(1)}(a_i^{(1)}) - \mu_i^{(1)}\big(b_i^{(1)}\big)\big) > \varepsilon - \frac{\varepsilon}{4},$$

$$P\big(\mu_j^{(1)}(a) - \mu_j^{(1)}(b)\big) < \frac{\varepsilon}{2^{i+3}}, a \leq a_i^{(1)}, b \leq b_i^{(1)}, j < i.$$

Let $(c_1^{(1)}, d_1^{(1)}) = (a_1^{(1)}, b_1^{(1)})$. Similarly, we can obtain a $(c_{k_1}^{(1)}, d_{k_1}^{(1)})$ and an i_{k_1} such that for all $j > i_{k_1}$, $P(\mu_j^{(1)}(c_{k_1}^{(1)} \wedge a_j^{(1)}) - \mu_j^{(1)}(d_{k_1}^{(1)} \wedge b_j^{(1)})) < \frac{\varepsilon}{8}$.

Let $p_2 = i_{k_1}$, $(h_2, g_2) = (c_{k_1}^{(1)}, d_{k_1}^{(1)})$, $\mu_i^{(2)} = \mu_{p_2+i}^{(1)}$, $(a_i^{(2)}, b_i^{(2)}) = (a_{p_2+i}^{(1)} \wedge h'_2, b_{p_2+i}^{(1)} \wedge g'_2)$. Then $h_1 \wedge h_2 = 0$, $g_1 \wedge g_2 = 0$, and $P(\mu_2(h_2) - \mu_2(g_2)) < \frac{\varepsilon}{32}$, $P(\mu_1^{(1)}(h_2) - \mu_1^{(1)}(g_2)) > \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{8}$. So,

$$P\big(\big(\mu_1^{(1)} - \mu_2\big)(h_2) - \big(\mu_1^{(1)} - \mu_2\big)(g_2)) > \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{8} - \frac{\varepsilon}{32},$$

$$P\big(\mu_i^{(2)}\big(a_i^{(2)}\big) - \mu_i^{(2)}\big(b_i^{(2)}\big)\big) > \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{8},$$

$$P\big(\mu_j^{(2)}(a) - \mu_j^{(2)}(b)\big) < \frac{\varepsilon}{2^{i+4}},\ a \leq a_i^{(2)},\ b \leq b_i^{(2)},\ j < i.$$

Inductively, we can obtain disjoint sequence $\{h_k\}$ and disjoint sequence $\{g_k\}$ of L, and a sequence of $\{\mu_1^{(k)}\}$ such that $P((\mu_1^{(k+1)} - \mu_1^{(k)})(h_{k+2}) - (\mu_1^{(k+1)} - \mu_1^{(k)})(g_{k+2})) > \varepsilon - \frac{\varepsilon}{4} - \frac{\varepsilon}{8} - \cdots - \frac{\varepsilon}{2^{k+1}} - \frac{\varepsilon}{32} > \frac{\varepsilon}{16}$ for all $k \in \mathbf{N}$.

This contradicts Theorem 1 of Junde and Zhihao (2003) again, so the theorem is proved. □

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